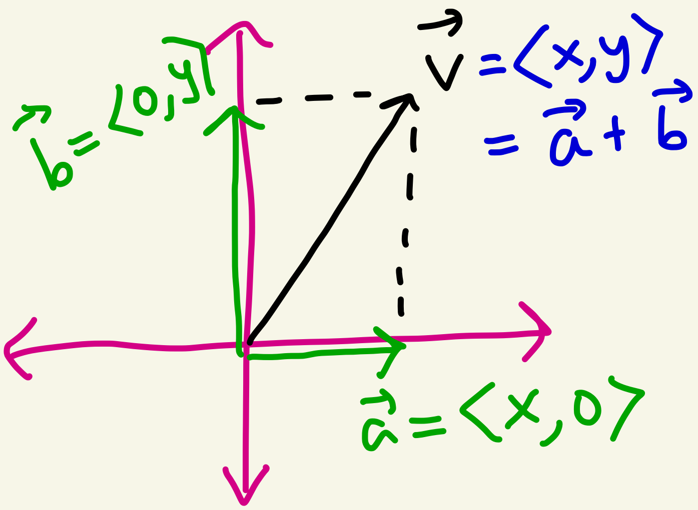
Topic 7 -

Spanning, linear independence, bases

HW 7 TOPIC — Spanning, Linear Independence, and Bases

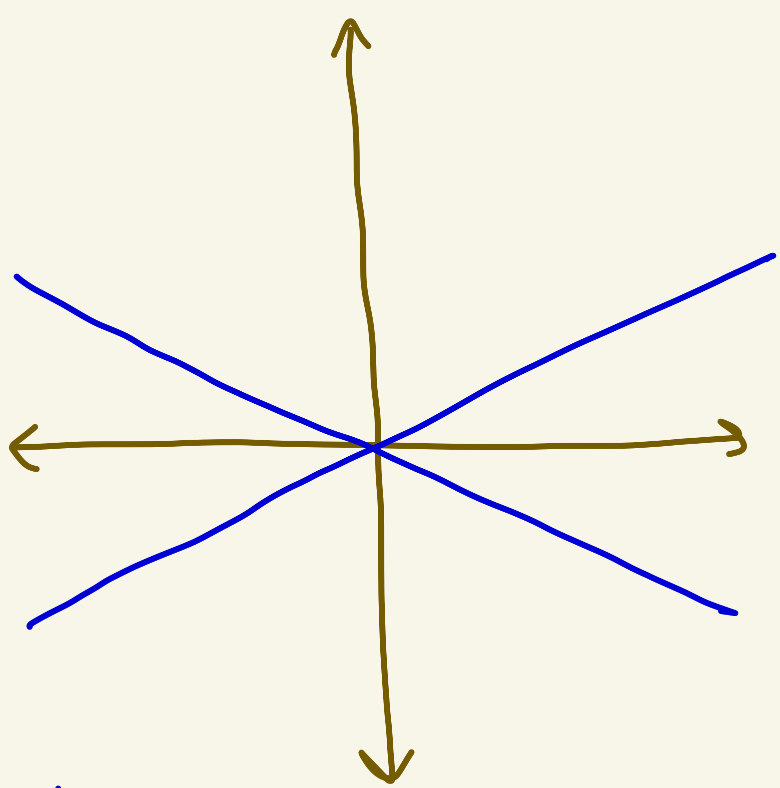
We are going to develop a way to create coordinate systems in vector spaces. This is what a basis will do.

$V = \mathbb{R}^2$



This is the x-axis/ y-axis coordinate system

other coordinate system



two other axes

<u>Def:</u> Let V be a vector space over a field F.

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be in V.

1. The <u>span</u> of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is defined to be the set

$$\operatorname{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}) = \Big\{ \underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n}_{\text{this is called a } \underline{\text{linear combination}} \text{ of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n} \;\Big|\; c_1, c_2, \ldots, c_n \in \mathbb{R} \Big\}$$

2. If $W = \operatorname{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\})$ then we say that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ <u>span</u> W.

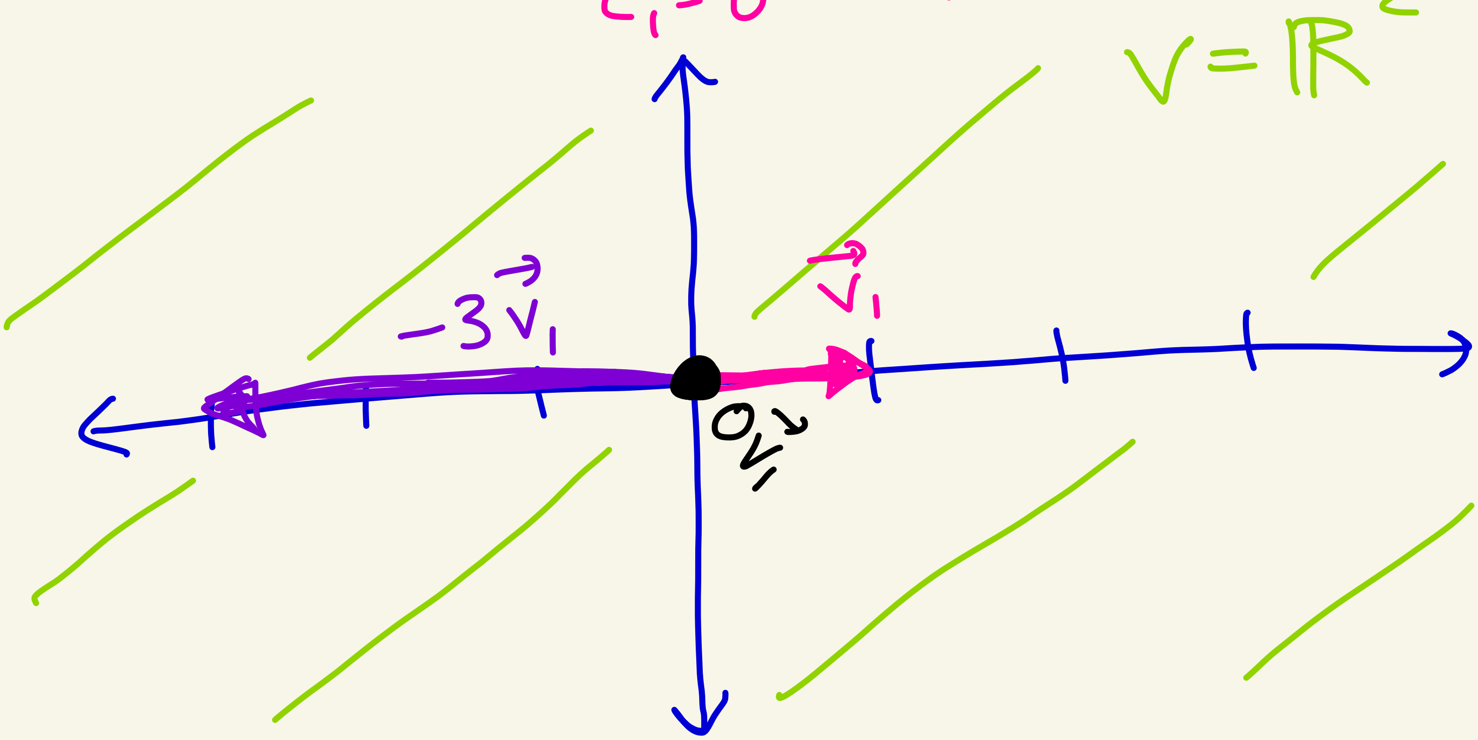
Ex: Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$

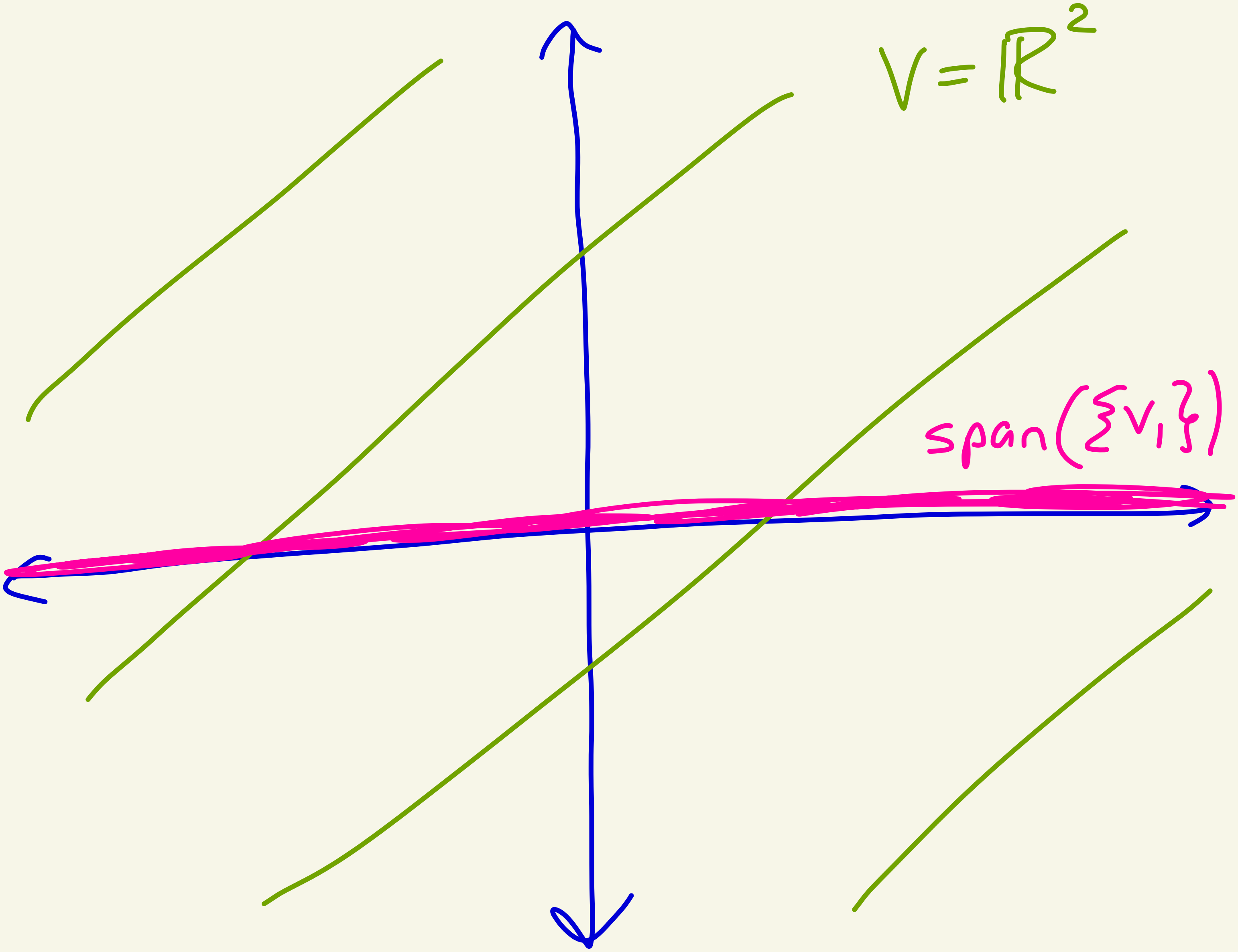
Let $\vec{v}_1 = \langle 1, 0 \rangle$ $\leftarrow$ $\boxed{\vec{v}_1 \text{ is in } \mathbb{R}^2}$

Then,

$$\text{span}(\{\vec{v}_1\}) = \{ c_1 v_1 \mid c_1 \in \mathbb{R} \}$$
$$= \{ c_1 \langle 1, 0 \rangle \mid c_1 \in \mathbb{R} \}$$
$$= \{ \langle c_1, 0 \rangle \mid c_1 \in \mathbb{R} \}$$
$$= \{ \underbrace{\langle 0, 0 \rangle}_{c_1 = 0}, \underbrace{\langle -3, 0 \rangle}_{c_1 = -3}, \underbrace{\langle 1, 0 \rangle}_{c_1 = 1}, \ldots \}$$

$\boxed{\text{x-axis}}$




$V = \mathbb{R}^2$
$\text{span}(\{v_1\})$

Ex: Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$.

Let $\vec{v}_1 = \langle 1, 0 \rangle$ and $\vec{v}_2 = \langle 0, 1 \rangle$

Then,

$$\text{span}(\{\vec{v}_1, \vec{v}_2\}) = \{c_1\vec{v}_1 + c_2\vec{v}_2 \mid c_1, c_2 \in \mathbb{R}\}$$

$$= \{c_1\langle 1, 0 \rangle + c_2\langle 0, 1 \rangle \mid c_1, c_2 \in \mathbb{R}\}$$

For example,

$$\underbrace{5}_{c_1 = 5} \cdot \langle 1, 0 \rangle - \underbrace{\pi}_{c_2 = -\pi} \cdot \langle 0, 1 \rangle = \langle 5, 0 \rangle + \langle 0, -\pi \rangle$$

$$= \langle 5, -\pi \rangle$$

is in the span of v_1, v_2.

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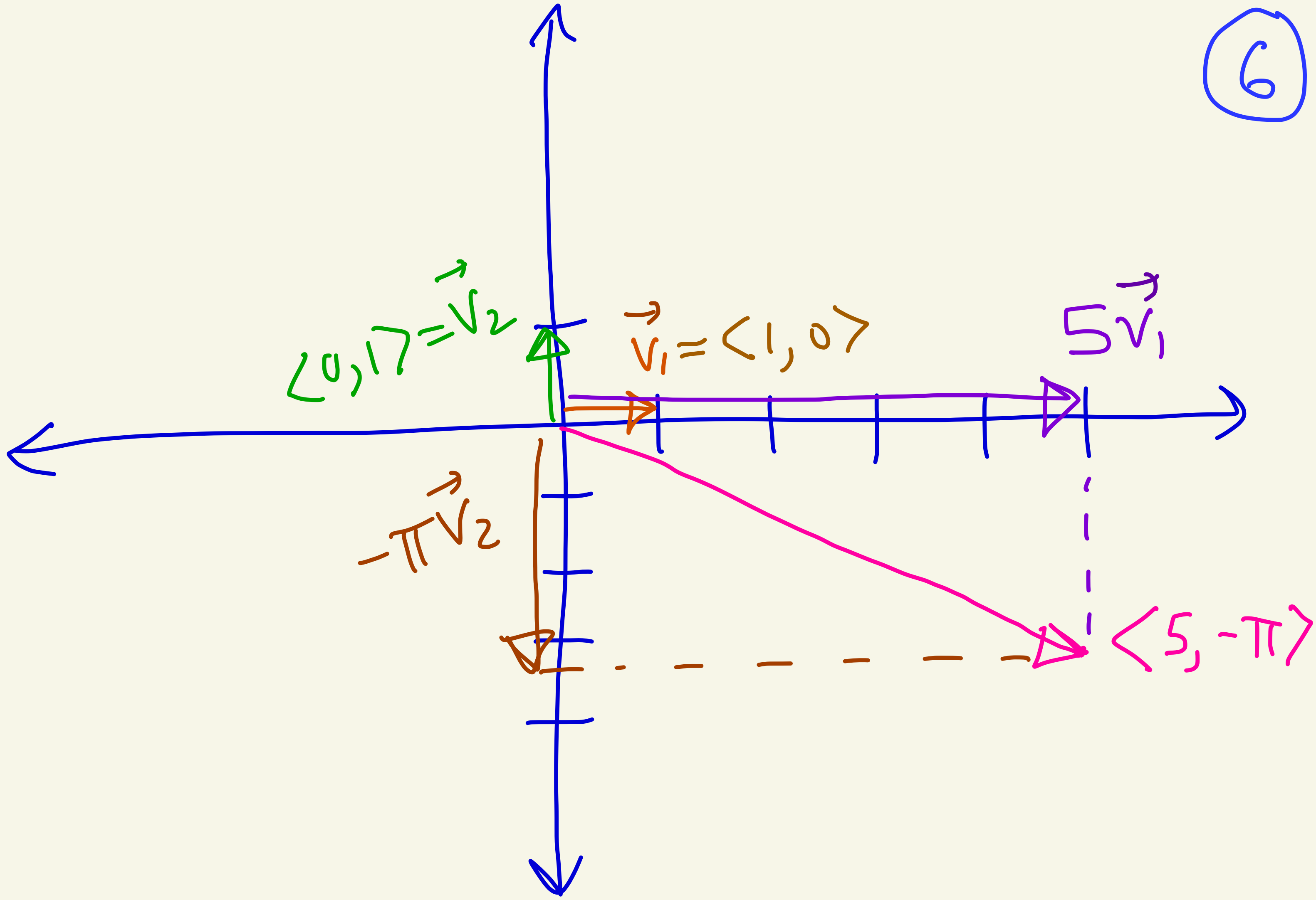

$$\langle 5, -\pi\rangle = \underbrace{5\vec{v}_1 - \pi\vec{v}_2}_{\text{in the span of } v_1, v_2}$$

Is $\langle 0,0 \rangle$ in $\text{span}(\{\vec{v}_1, \vec{v}_2\})$?

Yes, because

$$\langle 0,0 \rangle = 0 \cdot \underbrace{\langle 1,0 \rangle}_{\vec{v}_1} + 0 \cdot \underbrace{\langle 0,1 \rangle}_{\vec{v}_2}$$

Is $\langle -3, \frac{1}{2} \rangle$ in $\text{span}(\{\vec{v}_1, \vec{v}_2\})$?

Yes, because

$$\langle -3, \tfrac{1}{2} \rangle = -3 \cdot \underbrace{\langle 1,0 \rangle}_{\vec{v}_1} + \frac{1}{2} \cdot \underbrace{\langle 0,1 \rangle}_{\vec{v}_2}$$

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We have

$$\begin{aligned}
\operatorname{span}(\{\vec{v}_1, \vec{v}_2\}) &= \{c_1\langle 1,0\rangle + c_2\langle 0,1\rangle \mid c_1, c_2 \in \mathbb{R}\} \\
&= \{\langle c_1, 0\rangle + \langle 0, c_2\rangle \mid c_1, c_2 \in \mathbb{R}\} \\
&= \{\langle c_1, c_2\rangle \mid c_1, c_2 \in \mathbb{R}\} \\
&= \mathbb{R}^2 = V
\end{aligned}$$

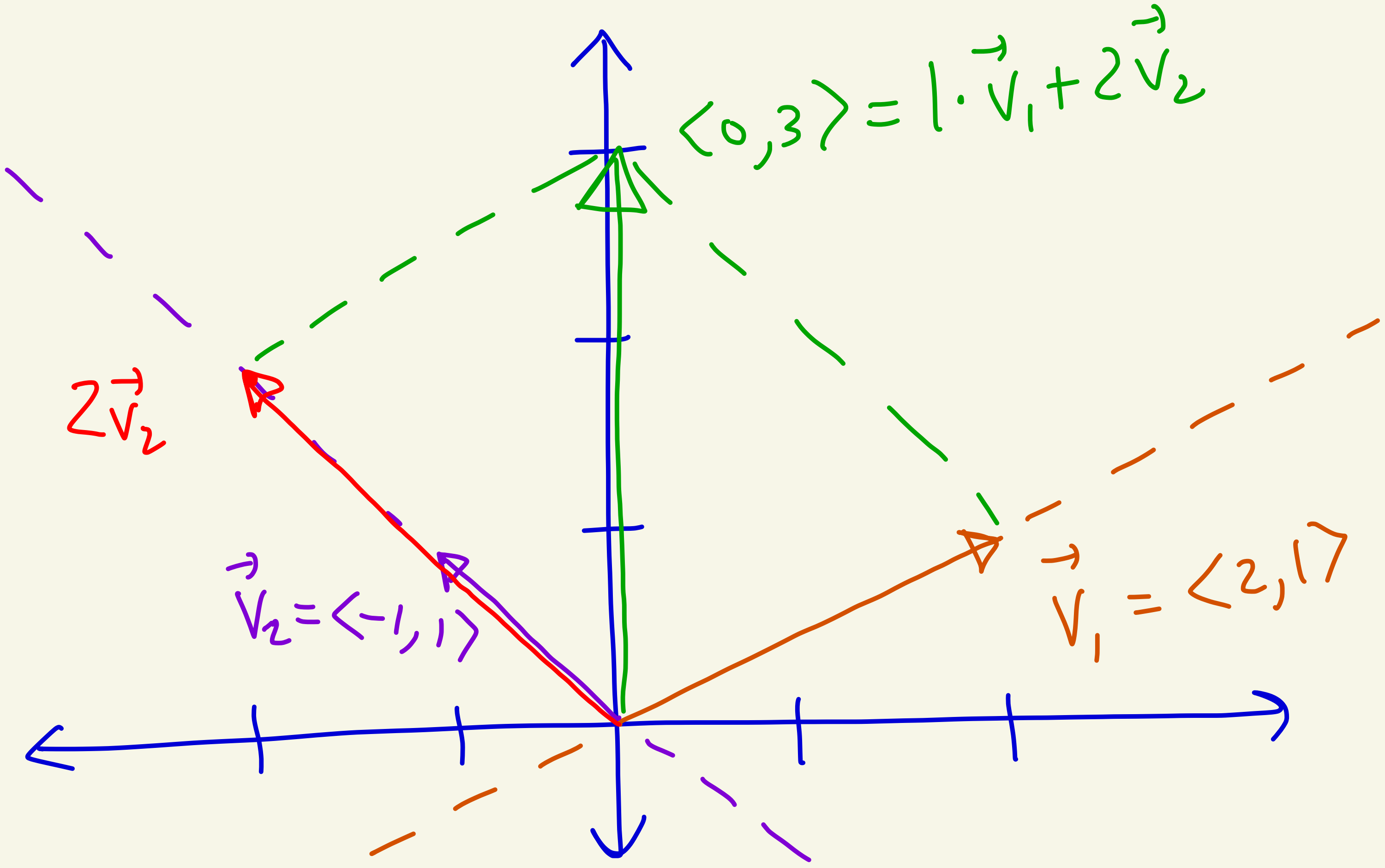
So, $\vec{v}_1 = \langle 1, 0\rangle$, $\vec{v}_2 = \langle 0, 1\rangle$ span $V = \mathbb{R}^2$.

<u>Ex:</u> Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$

Let $\vec{v}_1 = \langle 2, 1 \rangle$, $\vec{v}_2 = \langle -1, 1 \rangle$.

Let's list some vectors in $\operatorname{span}(\{\vec{v}_1, \vec{v}_2\})$.

$$\begin{aligned} 1 \cdot \vec{v}_1 + 2 \cdot \vec{v}_2 &= \langle 2, 1 \rangle + 2 \cdot \langle -1, 1 \rangle \\ &= \langle 2, 1 \rangle + \langle -2, 2 \rangle \\ &= \langle 0, 3 \rangle \end{aligned}$$



$$\frac{1}{2}\cdot\vec{v}_1 - \frac{3}{2}\vec{v}_2 = \frac{1}{2}\langle 2,1\rangle - \frac{3}{2}\langle -1,1\rangle$$

$$= \langle 1, \tfrac{1}{2}\rangle + \langle \tfrac{3}{2}, -\tfrac{3}{2}\rangle$$

$$= \langle \tfrac{5}{2}, -\tfrac{1}{2}\rangle$$

So, $\langle \frac{5}{2}, -\frac{1}{2}\rangle$ is in the span of $\vec{v}_1, \vec{v}_2$.

Claim: $\operatorname{span}(\{\vec{v}_1, \vec{v}_2\}) = \mathbb{R}^2$

proof of claim: Let $\langle x,y\rangle$ be in $\mathbb{R}^2$. We need to show that we can always solve

$$\langle x,y\rangle = c_1\vec{v}_1 + c_2\vec{v}_2$$

for c_1, c_2.

Let's solve

$$\langle x, y \rangle = c_1 \vec{v}_1 + c_2 \vec{v}_2 .$$

This becomes

$$\langle x, y \rangle = c_1 \langle 2, 1 \rangle + c_2 \langle -1, 1 \rangle$$

which becomes

$$\langle x, y \rangle = \langle 2c_1, c_1 \rangle + \langle -c_2, c_2 \rangle$$

which becomes

$$\langle x, y \rangle = \langle 2c_1 - c_2,\ c_1 + c_2 \rangle$$

which is equivalent to

$$\boxed{\begin{aligned} x &= 2c_1 - c_2 \\ y &= c_1 + c_2 \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} 2c_1 - c_2 &= x \\ c_1 + c_2 &= y \end{aligned}}$$

This system becomes

$$\left(\begin{array}{cc|c} 2 & -1 & x \\ 1 & 1 & y \end{array}\right) \xleftrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & y \\ 2 & -1 & x \end{array}\right)$$

$$\xrightarrow{-2R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & y \\ 0 & -3 & x-2y \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3} R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & y \\ 0 & 1 & -\frac{1}{3}x+\frac{2}{3}y \end{array}\right)$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 &= y \\ c_2 &= -\frac{1}{3}x + \frac{2}{3}y \end{aligned}} \quad \begin{array}{l} ① \\ ② \end{array}$$

$$c_2 = -\frac{1}{3}x + \frac{2}{3}y$$

$$\begin{aligned} c_1 &= y - c_2 \\ &= y - \left(-\frac{1}{3}x + \frac{2}{3}y\right) \\ &= \frac{1}{3}x + \frac{1}{3}y \end{aligned}$$

Thus, given any $\langle x,y\rangle$ in $\mathbb{R}^2$ we can write

$$\langle x,y\rangle = \underbrace{\left(\tfrac{1}{3}x+\tfrac{1}{3}y\right)\langle 2,1\rangle + \left(-\tfrac{1}{3}x+\tfrac{2}{3}y\right)\langle -1,1\rangle}_{c_1 \vec{v}_1 + c_2 \vec{v}_2}$$

For example, if $\langle x,y\rangle = \langle 12,-3\rangle$ then

$$\langle 12,-3\rangle = \left(\tfrac{1}{3}\cdot 12+\tfrac{1}{3}(-3)\right)\langle 2,1\rangle + \left(-\tfrac{1}{3}\cdot 12+\tfrac{2}{3}(-3)\right)\langle -1,1\rangle$$

$$= 3\langle 2,1\rangle - 6\langle -1,1\rangle$$

We showed any vector $\langle x,y\rangle$ in $\mathbb{R}^2$ is in the span of $\vec{v}_1 = \langle 2,1\rangle$, $\vec{v}_2 = \langle -1,1\rangle$.

Thus, $\vec{v}_1 = \langle 2,1\rangle$, $\vec{v}_2 = \langle -1,1\rangle$ span $\mathbb{R}^2$

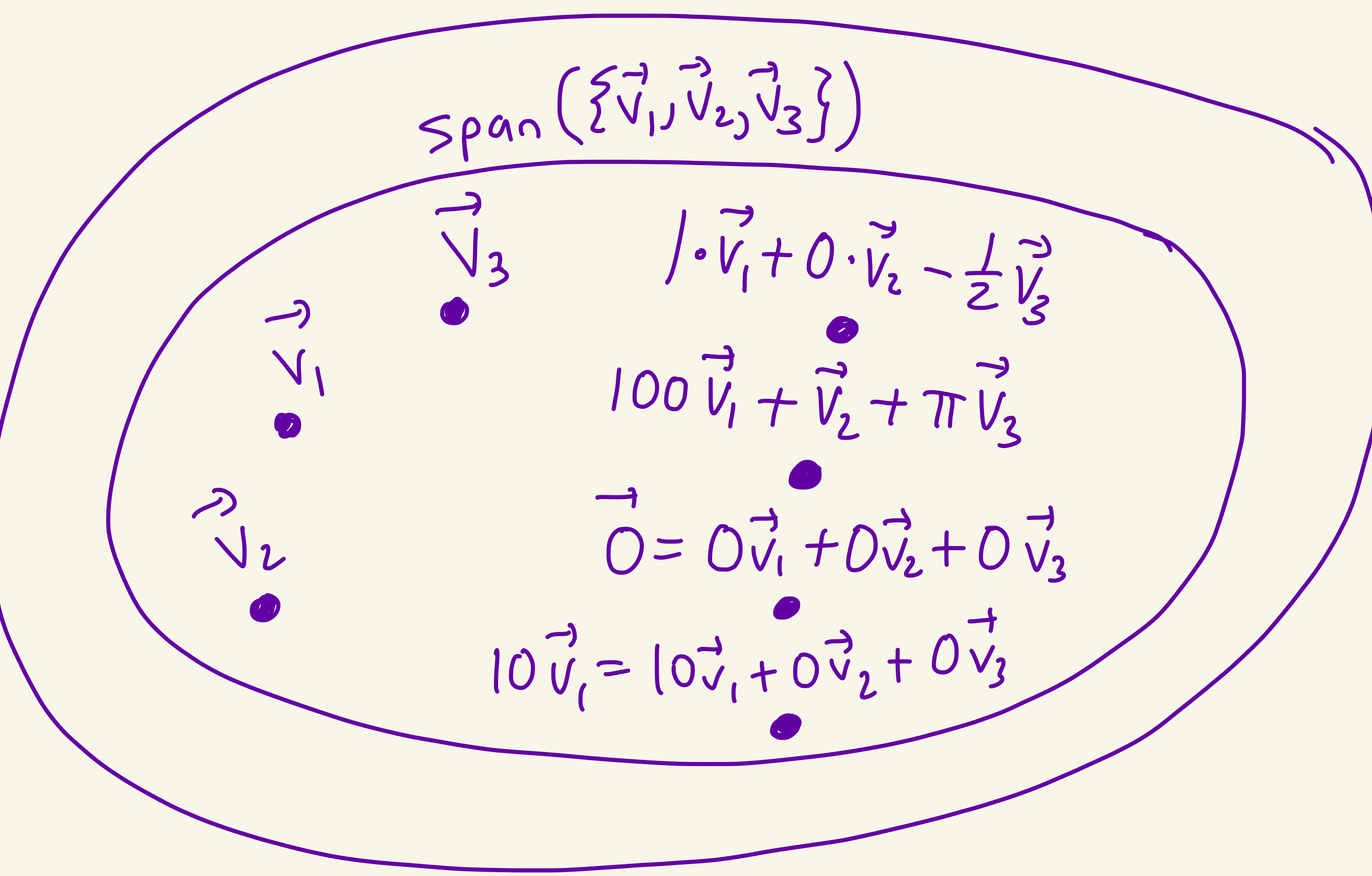
or you can write $\operatorname{span}(\{\vec{v}_1,\vec{v}_2\}) = \mathbb{R}^2$.

Theorem: Let V be a vector space over a field F. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be in V. Then $\text{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\})$ is a subspace of V.

So we create subspaces of V by picking some vectors and their creating span

picture when $n=3$

V



Def: Let V be a vector space over a field F.
Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be in V.
We say that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are <u>linearly dependent</u> if there exist scalars $c_1, c_2, \ldots, c_n$ from F that are not all equal to zero (but some can be zero) such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are not linearly dependent, then we say that they are <u>linearly independent</u>.

Ex: Let $V = \mathbb{R}^3$, $F = \mathbb{R}$.

Let $\vec{v}_1 = \langle 1, 1, 2 \rangle$ and $\vec{v}_2 = \langle -2, -2, -4 \rangle$

Note that $\vec{v}_2 = -2\vec{v}_1$

So, $\boxed{2 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 = \vec{0}}$

Thus, $c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$ has the solution $c_1 = 2$, $c_2 = 1$ and c_1, c_2 are not both equal to zero.

Thus, $\vec{v}_1 = \langle 1, 1, 2 \rangle$, $\vec{v}_2 = \langle -2, -2, -4 \rangle$ are linearly dependent.

Ex: Let $V = \mathbb{R}^3$ and $F = \mathbb{R}$.

Let $\vec{v}_1 = \langle 1,1,1 \rangle$, $\vec{v}_2 = \langle 1,0,1 \rangle$,
$\vec{v}_3 = \langle 1, \frac{4}{3}, 1 \rangle$

Are these vectors linearly dependent or linearly independent?

Consider the equation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

This becomes

$$c_1 \langle 1,1,1 \rangle + c_2 \langle 1,0,1 \rangle + c_3 \langle 1, \tfrac{4}{3}, 1 \rangle = \langle 0,0,0 \rangle$$

This becomes

$$\langle c_1, c_1, c_1 \rangle + \langle c_2, 0, c_2 \rangle + \langle c_3, \tfrac{4}{3}c_3, c_3 \rangle = \langle 0,0,0 \rangle$$

This becomes

$$\langle c_1 + c_2 + c_3,\ c_1 + \tfrac{4}{3}c_3,\ c_1 + c_2 + c_3 \rangle = \langle 0,0,0 \rangle$$

We had

$$\langle c_1+c_2+c_3,\ c_1+\tfrac{4}{3}c_3,\ c_1+c_2+c_3\rangle = \langle 0,0,0\rangle$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 \quad\;\; + \tfrac{4}{3}c_3 &= 0 \\ c_1 + c_2 + c_3 &= 0 \end{aligned}}$$

Let's solve this system:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & 4/3 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow[-R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This becomes

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 0 \quad (1)\\ c_2 - \tfrac{1}{3}c_3 &= 0 \quad (2)\\ 0 &= 0 \quad (3)\end{aligned}}$$

c_1, c_2 are leading variables.
c_3 is free variable

$$\boxed{\begin{aligned} c_1 &= -c_2 - c_3 \quad (1)\\ c_2 &= 1/3\, c_3 \quad (2)\\ 0 &= 0 \quad (3)\end{aligned}}$$

Set $c_3 = t$.

(2) gives $c_2 = \frac{1}{3}t$

(1) gives $c_1 = -c_2 - c_3 = -\left(\frac{1}{3}t\right) - t = -\frac{4}{3}t$

Solutions are:

$c_1 = -\frac{4}{3}t$

$c_2 = \frac{1}{3}t$

$c_3 = t$

where t can be any real number

Thus the solutions to

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

are $c_1 = -\frac{4}{3}t$, $c_2 = \frac{1}{3}t$, $c_3 = t$ where t is any real number.

Thus, for any real number t we have that

$$\underbrace{\left(-\tfrac{4}{3}t\right)}_{c_1}\underbrace{\langle 1,1,1\rangle}_{\vec{v}_1} + \underbrace{\left(\tfrac{1}{3}t\right)}_{c_2}\underbrace{\langle 1,0,1\rangle}_{\vec{v}_2} + \underbrace{t}_{c_3}\underbrace{\langle 1,\tfrac{4}{3},1\rangle}_{\vec{v}_3} = \vec{0}$$

For example if $t=1$, then

$$-\frac{4}{3}\langle 1,1,1\rangle + \frac{1}{3}\langle 1,0,1\rangle + 1\cdot\langle 1,\tfrac{4}{3},1\rangle = \vec{0}$$

So,

$$\boxed{-\frac{4}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_2 + 1\cdot\vec{v}_3 = \vec{0}}$$

Thus, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

Ex: Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$.

Let $\vec{v}_1 = \langle 1, 0 \rangle$ and $\vec{v}_2 = \langle 0, 1 \rangle$.

Are $\vec{v}_1, \vec{v}_2$ linearly independent or linearly dependent?

Consider the equation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$$

This becomes

$$c_1 \langle 1, 0 \rangle + c_2 \langle 0, 1 \rangle = \langle 0, 0 \rangle$$

This gives

$$\langle c_1, 0 \rangle + \langle 0, c_2 \rangle = \langle 0, 0 \rangle$$

This gives

$$\langle c_1, c_2 \rangle = \langle 0, 0 \rangle$$

Thus, $c_1 = 0$, $c_2 = 0$.

Thus, the only solution to

$$\underbrace{c_1\langle 1,0\rangle + c_2\langle 0,1\rangle = \langle 0,0\rangle}_{c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}}$$

is $c_1 = c_2 = 0$.

Thus, $\vec{v}_1 = \langle 1,0\rangle$, $\vec{v}_2 = \langle 0,1\rangle$ are linearly independent.

Ex: Let

$$V = P_2 = \{ a+bx+cx^2 \mid a,b,c \in \mathbb{R} \}$$

$$F = \mathbb{R}$$

Let $\vec{v}_1 = 1$

$\vec{v}_2 = 1 + x$

$\vec{v}_3 = 1 + x + x^2$

Are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent or linearly dependent?

Consider the equation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

If there is only one solution $c_1 = c_2 = c_3 = 0$ then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

If there are more solutions they are linearly dependent.

This equation becomes

$$c_1\underset{\substack{\uparrow\\ 1+0x+0x^2}}{(1)} + c_2\underset{\substack{\uparrow\\ 1+x+0x^2}}{(1+x)} + c_3(1+x+x^2) = \underbrace{0+0x+0x^2}_{\vec{0}}$$

This becomes

$$c_1 + c_2 + c_2x + c_3 + c_3x + c_3x^2 = 0+0x+0x^2$$

This gives

$$(c_1+c_2+c_3) + (c_2+c_3)x + c_3x^2 = 0+0x+0x^2$$

This gives

$$\boxed{\begin{aligned} c_1+c_2+c_3 &= 0 \quad (1)\\ c_2+c_3 &= 0 \quad (2)\\ c_3 &= 0 \quad (3)\end{aligned}}$$

This is already reduced.

Solution is:

(3) gives $c_3 = 0$

(2) gives $c_2 = -c_3 = 0$

(1) gives $c_1 = -c_2 - c_3$
$= 0+0 = 0$

Since the only solution to

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

is $c_1 = c_2 = c_3 = 0$,

this means that

$$\vec{v}_1 = 1$$
$$\vec{v}_2 = 1 + x$$
$$\vec{v}_3 = 1 + x + x^2$$

are linearly independent.

We will now create the idea of a coordinate system in a vector space. Its called a basis.

Def: Let V be a vector space over a field F.
Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be in V.
We say that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are a **basis** for V if

(1) $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span V

and

(2) $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent

Idea: (1) makes it so that every vector $\vec{v}$ in V can be written in the form $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$

(2) makes it so that there is only one way to write $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$
ie the constants are unique to $\vec{v}$

Ex: Let $V=\mathbb{R}^2$, $F=\mathbb{R}$.

Let $\vec{v}_1 = \langle 1,0 \rangle$, $\vec{v}_2 = \langle 0,1 \rangle$.

In class we showed $\vec{v}_1, \vec{v}_2$ span $\mathbb{R}^2$.

Let's do this again.

Given $\langle x,y \rangle$ in $V=\mathbb{R}^2$ we can write

$$\underbrace{\langle x,y \rangle = x\langle 1,0 \rangle + y\langle 0,1 \rangle}$$

So, $\langle x,y \rangle$ is in the span of $\vec{v}_1 = \langle 1,0 \rangle$, $\vec{v}_2 = \langle 0,1 \rangle$.

We just showed that $\vec{v}_1, \vec{v}_2$ are linearly independent.

Thus, $\vec{v}_1 = \langle 1,0 \rangle$, $\vec{v}_2 = \langle 0,1 \rangle$ is a basis for $V=\mathbb{R}^2$.

This is called the <u>standard basis</u>.

<u>Ex:</u> Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$.

Let $\vec{v}_1 = \langle 2, 1 \rangle$ and $\vec{v}_2 = \langle -1, 1 \rangle$

Earlier, we showed $\vec{v}_1, \vec{v}_2$ span $\mathbb{R}^2$, in particular we showed that if $\langle x, y \rangle$ is in $\mathbb{R}^2$ then

$$\langle x, y \rangle = \underbrace{\left(\tfrac{1}{3}x + \tfrac{1}{3}y\right)\vec{v}_1 + \left(-\tfrac{1}{3}x + \tfrac{2}{3}y\right)\vec{v}_2}_{c_1\vec{v}_1 + c_2\vec{v}_2}$$

Now we will show that $\vec{v}_1, \vec{v}_2$ are a basis for $\mathbb{R}^2$. We just need to show that $\vec{v}_1, \vec{v}_2$ are linearly independent.

Suppose we have

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$$

We can always write $0\vec{v}_1 + 0\vec{v}_2 = \vec{0}$. If thats the only sol. then $\vec{v}_1, \vec{v}_2$ are lin. ind. If there are more ways to express $\vec{0}$ in terms of $\vec{v}_1, \vec{v}_2$ then $\vec{v}_1, \vec{v}_2$ are lin. dep.

What are the solutions in terms of c_1, c_2 ?

We have

$$c_1 \underbrace{\langle 2,1 \rangle}_{\vec{v}_1} + c_2 \underbrace{\langle -1,1 \rangle}_{\vec{v}_2} = \underbrace{\langle 0,0 \rangle}_{\vec{0}}$$

This becomes

$$\langle 2c_1, c_1 \rangle + \langle -c_2, c_2 \rangle = \langle 0,0 \rangle$$

which becomes

$$\langle 2c_1 - c_2, c_1 + c_2 \rangle = \langle 0,0 \rangle$$

So we get

$$\boxed{\begin{aligned} 2c_1 - c_2 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}}$$

This gives

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -3 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 &= 0 \quad (1) \\ c_2 &= 0 \quad (2) \end{aligned}}$$

(2) gives $c_2 = 0$.

(1) gives $c_1 = -c_2 = -0 = 0$.

Thus, the only sol. to $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ is $c_1 = c_2 = 0$. So, $\vec{v}_1 = \langle 2, 1 \rangle$, $\vec{v}_2 = \langle -1, 1 \rangle$ are linearly independent and thus form a basis for $\mathbb{R}^2$.

Theorem: Let V be a vector space over a field F. Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a basis for V. Then any other basis will also have n elements in it.

Translation: Any two bases for V have the same number of elements in the basis.

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$

We found two bases for $\mathbb{R}^2$ so far:

basis #1: $\langle 1, 0 \rangle, \langle 0, 1 \rangle$ ← $\boxed{\text{standard basis for } \mathbb{R}^2}$

basis #2: $\langle 2, 1 \rangle, \langle -1, 1 \rangle$

What the theorem above says is that since we've found a basis for $\mathbb{R}^2$ with $n=2$ vectors in it, every basis for $\mathbb{R}^2$ will have 2 vectors in it.

Def: Let V be a vector space over a field F. If there exists a basis for V with n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then we say that V is finite-dimensional and the dimension of V is n.

We write $\dim(V) = n$.

some people write $\dim_F(V) = n$

Ex: Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$.

A basis for $\mathbb{R}^2$ is $\langle 1, 0 \rangle, \langle 0, 1 \rangle$.

There are 2 vectors in the basis, so $\mathbb{R}^2$ is finite-dimensional and $\dim(\mathbb{R}^2) = 2$.

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<u>Ex:</u> Let

$V = P_2 = \{ a + bx + cx^2 \mid a, b, c \in \mathbb{R} \}$

$F = \mathbb{R}$.

Let $\boxed{\vec{v}_1 = 1, \ \vec{v}_2 = x, \ \vec{v}_3 = x^2}$

<u>Claim:</u> $\vec{v}_1 = 1, \ \vec{v}_2 = x, \ \vec{v}_3 = x^2$ is a basis for P_2

<u>proof:</u>

① We first show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ span P_2.

Let $a + bx + cx^2$ be in P_2.

Then, $a + bx + cx^2 = a \cdot 1 + b \cdot x + c \cdot x^2$

$= a \cdot \vec{v}_1 + b \cdot \vec{v}_2 + c \cdot \vec{v}_3$

So, $P_2 = \operatorname{span}(\{1, x, x^2\})$

2 Now we show that

$\vec{v}_1 = 1$, $\vec{v}_2 = x$, $\vec{v}_3 = x^2$ are linearly independent.

Consider the equation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

How many solutions does the above equation have?

The above equation becomes

$$c_1 \cdot 1 + c_2 \cdot x + c_3 x^2 = \overbrace{0 \cdot 1 + 0 \cdot x + 0 \cdot x^2}^{\vec{0} \text{ in } P_2}$$

So, $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

Since $c_1 = c_2 = c_3 = 0$ is the only solution to $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$ we know $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent

We have shown that

$\vec{v}_1 = 1$, $\vec{v}_2 = x$, $\vec{v}_3 = x^2$

is a basis for P_2.

Thus, P_2 is finite-dimensional

and $\dim(P_2) = 3$

$\boxed{\text{Special example}}$:

The "trivial" vector space is the vector space $V = \{\vec{0}\}$ over a field F. So, V just has one vector.

There is no basis for this vector space.

We just define this special vector space to have dimension 0.

Ex: Let $V = \mathbb{R}^n$ and $F = \mathbb{R}$.

The <u>standard basis</u> for $\mathbb{R}^n$ is the set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ where $\vec{v}_i$ has a 1 in spot i and 0's everywhere else.

One can show that this is a basis for $\mathbb{R}^n$ and thus

$$\boxed{\dim(\mathbb{R}^n) = n.}$$

n	standard basis for $\mathbb{R}^n$
2	$\vec{v}_1 = \langle 1, 0 \rangle$, $\vec{v}_2 = \langle 0, 1 \rangle$
3	$\vec{v}_1 = \langle 1, 0, 0 \rangle$, $\vec{v}_2 = \langle 0, 1, 0 \rangle$, $\vec{v}_3 = \langle 0, 0, 1 \rangle$
4	$\vec{v}_1 = \langle 1, 0, 0, 0 \rangle$, $\vec{v}_2 = \langle 0, 1, 0, 0 \rangle$, $\vec{v}_3 = \langle 0, 0, 1, 0 \rangle$, $\vec{v}_4 = \langle 0, 0, 0, 1 \rangle$
$\vdots$	$\vdots$ $\vdots$

<u>Ex:</u> Let n be an integer with $n \geq 0$.

Then

$$P_n = \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_0, a_1, \ldots, a_n \in \mathbb{R} \}$$

The standard basis for P_n is

$$\vec{v}_0 = 1$$
$$\vec{v}_1 = x$$
$$\vec{v}_2 = x^2$$
$$\vdots$$
$$\vec{v}_n = x^n$$

You can show that $\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_n$ are a basis for P_n [we did that last time for $n=2$].

Thus, $\dim(P_n) = n+1$

n	Standard basis for P_n
0	1
1	$1, x$
2	$1, x, x^2$
3	$1, x, x^2, x^3$
4	$1, x, x^2, x^3, x^4$
$\vdots$	$\vdots$
n	$1, x, x^2, \ldots, x^n$

<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F with $\dim(V)=n$.

So, this means V has a basis with n vectors in it.

Let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ be in V.

① If $m<n$, then $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ do <u>not</u> span V.

② If $m>n$, then $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ are linearly dependent.

Ex: Let $V = \mathbb{R}^3$, $F = \mathbb{R}$.

We know the standard basis for $\mathbb{R}^3$ is $\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle$.

So, $\dim(\mathbb{R}^3) = 3$.

$\boxed{n = 3 \text{ in theorem on pg 4}}$

Let $\vec{w}_1 = \langle 1,1,1\rangle$, $\vec{w}_2 = \langle \pi, \frac{1}{2}, 3\rangle$.

$\boxed{m = 2 \text{ in theorem on page 4}}$

We have 2 vectors in a 3-dimensional space. Since $2 < 3$, by the previous theorem, $\vec{w}_1, \vec{w}_2$ do not span $\mathbb{R}^3$.

[Also, they aren't a basis since any basis for $\mathbb{R}^3$ must have 3 vectors in it.]

Ex: Let $V = P_3$, $F = \mathbb{R}$.

The standard basis for P_3 is $1, x, x^2, x^3$. Thus, $\dim(P_3) = 4$.

Let

$$\vec{w}_1 = 1 + 3x^2$$
$$\vec{w}_2 = 2x - 5x^3$$
$$\vec{w}_3 = 5$$
$$\vec{w}_4 = x^3$$
$$\vec{w}_5 = x^2 - x^3$$
$$\vec{w}_6 = 1 + x + x^2 + x^3$$

We have 6 vectors in a 4-dimensional vector space, thus the previous theorem says that $\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4, \vec{w}_5, \vec{w}_6$ are linearly dependent.

[Also, they are not a basis for P_3.]

<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F.

Suppose $\dim(V) = n$.

Suppose we pick n vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ from V.

1. If $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are linearly independent, then $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ span V and hence they form a basis for V.
2. If $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ span V, then $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are linearly independent and hence they form a basis for V.

Ex: Let $V = P_2$ and $F = \mathbb{R}$.

We know that $\dim(P_2) = 3$.

Let

$$\vec{v}_1 = 1$$

$$\vec{v}_2 = 1 + x$$

$$\vec{v}_3 = 1 + x + x^2$$

We saw earlier that $1, 1+x, 1+x+x^2$ are linearly independent.

Since we have 3 linearly independent vectors in a 3-dimensional space $V = P_2$, from the previous theorem we know that

$$1, \ 1+x, \ 1+x+x^2$$

form a basis for $V = P_2$.

Here is the point of having a basis. It gives a coordinate system for V.

Theorem: Let V be a vector space over a field F. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a basis for V. Then given any vector $\vec{x}$ in V there exist unique scalars $c_1, c_2, \ldots, c_n$ from F where

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n$$

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

We showed that
$\vec{v}_1 = \langle 2, 1 \rangle$, $\vec{v}_2 = \langle -1, 1 \rangle$
is a basis for $\mathbb{R}^2$.

Pick $\vec{x} = \langle 5, 8 \rangle$.

Because $\vec{v}_1, \vec{v}_2$ span $V = \mathbb{R}^2$
we know we can solve

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

By the previous theorem, there will be unique c_1, c_2 that solve the above.

Let's solve it. We have

$$\begin{pmatrix} 5 \\ 8 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

This becomes

$$\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 2c_1 - c_2 \\ c_1 + c_2 \end{pmatrix}$$

This gives

$$\boxed{\begin{aligned} 2c_1 - c_2 &= 5 \\ c_1 + c_2 &= 8 \end{aligned}}$$

$$\left(\begin{array}{cc|c} 2 & -1 & 5 \\ 1 & 1 & 8 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & 8 \\ 2 & -1 & 5 \end{array}\right)$$

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & -3 & -11 \end{array}\right) \xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & \frac{11}{3} \end{array}\right)$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 &= 8 \\ c_2 &= \tfrac{11}{3} \end{aligned}} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

(2) gives $c_2 = \frac{11}{3}$

(1) gives

$$\begin{aligned} c_1 &= 8 - c_2 \\ &= 8 - \frac{11}{3} \\ &= \frac{24 - 11}{3} \\ &= 13/3 \end{aligned}$$

Thus,

$$\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \frac{13}{3} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{11}{3} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \frac{13}{3} \cdot \vec{v}_1 + \frac{11}{3} \cdot \vec{v}_2$$

these numbers $13/3$, $11/3$ will be called the coordinates of $\vec{x} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$ in terms of the basis $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Def: Let V be a vector space over a field F. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a basis for V. If we fix this ordering on the basis, then we call this an <u>ordered basis</u> for V. We write $\beta = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$ to denote an ordered basis.

So the ordering matters here.

β is beta

β is the name we gave to the basis.

Given $\vec{x}$ in V we can write

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

where $c_1, c_2, \ldots, c_n$ are unique elements of F.

The constants $c_1, c_2, \ldots, c_n$ are called the <u>coordinates of $\vec{x}$ with respect to the ordered basis β</u>

and we write

$$[\vec{x}]_\beta = \langle c_1, c_2, \ldots, c_n \rangle$$

or

$$[\vec{x}]_\beta = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Ex: Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$.

We know $\langle 1,0\rangle, \langle 0,1\rangle$ is a basis for $\mathbb{R}^2$ and $\langle 2,1\rangle, \langle -1,1\rangle$ is also a basis for $\mathbb{R}^2$.

Let $\beta = [\langle 1,0\rangle, \langle 0,1\rangle]$

and $\gamma = [\langle 2,1\rangle, \langle -1,1\rangle]$

and $\beta' = [\langle 0,1\rangle, \langle 1,0\rangle]$

same but order changed

This gives us three different ordered bases for $\mathbb{R}^2$.

Let's look at $\vec{x} = \langle 5,8\rangle$.

We have

$$\vec{x} = \langle 5,8 \rangle = 5 \cdot \langle 1,0 \rangle + 8 \cdot \langle 0,1 \rangle$$

Thus,

$$[\vec{x}]_\beta = \langle 5,8 \rangle$$

$$\boxed{\beta = [\langle 1,0 \rangle, \langle 0,1 \rangle]}$$

We also have

$$\vec{x} = \langle 5,8 \rangle = 8 \cdot \langle 0,1 \rangle + 5 \cdot \langle 1,0 \rangle$$

Thus,

$$[\vec{x}]_{\beta'} = \langle 8, 5 \rangle$$

$$\boxed{\beta' = [\langle 0,1 \rangle, \langle 1,0 \rangle]}$$

Also, from last week we saw

$$\vec{x} = \langle 5,8 \rangle = \frac{13}{3} \cdot \langle 2,1 \rangle + \frac{11}{3} \cdot \langle -1,1 \rangle$$

Thus,

$$[\vec{x}]_\gamma = \left\langle \frac{13}{3}, \frac{11}{3} \right\rangle$$

$$\gamma = [\langle 2,1 \rangle, \langle -1,1 \rangle]$$

Ex: Let

$V = P_2 = \{ a+bx+cx^2 \mid a,b,c \in \mathbb{R} \}$

and $F = \mathbb{R}$.

Standard basis for P_2 is $1, x, x^2$.

In HW 7 - Part 1 problem 8(a) you show that $1,\ 1+x,\ 1+x+x^2$ is also a basis for P_2

Let $\beta = [1, x, x^2]$ and

$\gamma = [1,\ 1+x,\ 1+x+x^2]$.

Let $\vec{v} = 4+2x+3x^2$.

Let's find $[\vec{v}]_\beta$ and $[\vec{v}]_\gamma$.

We have that

$$\vec{v} = 4+2x+3x^2 = 4\cdot 1 + 2\cdot x + 3\cdot x^2$$

$$\beta = [1, x, x^2]$$

Thus,

$$[\vec{v}]_\beta = \langle 4, 2, 3 \rangle$$

Let's now find $[\vec{v}]_\gamma$.

We need to solve

$$\underbrace{4+2x+3x^2}_{\vec{v}} = \underbrace{c_1(1) + c_2(1+x) + c_3(1+x+x^2)}_{\gamma = [1,\, 1+x,\, 1+x+x^2]}$$

This becomes

$$4+2x+3x^2 = (c_1+c_2+c_3) + (c_2+c_3)x + c_3 x^2$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 4 \quad (1) \\ c_2 + c_3 &= 2 \quad (2) \\ c_3 &= 3 \quad (3) \end{aligned}}$$

This is already reduced

(3) gives $c_3 = 3$.

(2) gives $c_2 = 2 - c_3 = 2 - 3 = -1$

(1) gives $c_1 = 4 - c_2 - c_3 = 4 - (-1) - 3 = 2$

Thus,

$$\underbrace{4 + 2x + 3x^2}_{\vec{v}} = 2 \cdot (1) - 1 \cdot (1 + x) + 3 \cdot (1 + x + x^2)$$

So,

$$[\vec{v}]_\gamma = \langle 2, -1, 3 \rangle$$

Keeping $\gamma = [1, 1+x, 1+x+x^2]$, if you know that $[\vec{w}]_\gamma = \langle -1, 2, -3 \rangle$, what is $\vec{w}$?

We have that

$$\vec{w} = -1 \cdot (1) + 2(1+x) - 3 \cdot (1+x+x^2)$$

$$= -2 - x - 3x^2$$

Ex: Let

$$V = M_{2,2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R} \right\}$$

$F = \mathbb{R}$.

Claim: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a basis for $M_{2,2}$ and thus $\dim(M_{2,2}) = 4$. [This is called the standard basis for $M_{2,2}$]

Proof of claim:

① First we will show they span $M_{2,2}$.

Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from $M_{2,2}$ we have

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \\ &= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

Thus, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ span $M_{2,2}$.

② Let's now check linear independence.

Consider

$$c_1\begin{pmatrix}1&0\\0&0\end{pmatrix}+c_2\begin{pmatrix}0&1\\0&0\end{pmatrix}+c_3\begin{pmatrix}0&0\\1&0\end{pmatrix}+c_4\begin{pmatrix}0&0\\0&1\end{pmatrix}=\underbrace{\begin{pmatrix}0&0\\0&0\end{pmatrix}}_{\vec{0}}$$

This becomes

$$\begin{pmatrix}c_1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&c_2\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\c_3&0\end{pmatrix}+\begin{pmatrix}0&0\\0&c_4\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

which becomes

$$\begin{pmatrix}c_1&c_2\\c_3&c_4\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Thus, $c_1=0,\ c_2=0,\ c_3=0,\ c_4=0.$

Since we only got one solution the vectors $\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}$ are linearly independent.

By ① and ②, $\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}$ form a basis for $M_{2,2}$. and $\dim(M_{2,2})=4$ $\blacksquare$

Hw 7 - Part 1 #10(a)

Show that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is a basis for $M_{2,2}$

We already know $\dim(M_{2,2}) = 4$ from our previous example. And we have 4 vectors above. So if we show that they are linearly independent then by a theorem from class they will also span $M_{2,2}$ and thus be a basis for $M_{2,2}$.

Consider

$$c_1\begin{pmatrix}1&0\\0&1\end{pmatrix}+c_2\begin{pmatrix}1&1\\1&0\end{pmatrix}+c_3\begin{pmatrix}0&0\\0&1\end{pmatrix}+c_4\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\underbrace{\begin{pmatrix}0&0\\0&0\end{pmatrix}}_{\vec{0}}$$

This becomes

$$\begin{pmatrix}c_1&0\\0&c_1\end{pmatrix}+\begin{pmatrix}c_2&c_2\\c_2&0\end{pmatrix}+\begin{pmatrix}0&0\\0&c_3\end{pmatrix}+\begin{pmatrix}0&-c_4\\c_4&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

which gives

$$\begin{pmatrix}c_1+c_2&c_2-c_4\\c_2+c_4&c_1+c_3\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

This gives

$$\boxed{\begin{aligned}c_1+c_2&=0\\c_2-c_4&=0\\c_2+c_4&=0\\c_1+c_3&=0\end{aligned}}$$

We have

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{-R_1+R_4 \to R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_2+R_4 \to R_4]{-R_2+R_3 \to R_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_4 \to R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Thus we get

$$\boxed{\begin{aligned} c_1 + c_2 &= 0 \quad (1) \\ c_2 - c_4 &= 0 \quad (2) \\ c_3 - c_4 &= 0 \quad (3) \\ c_4 &= 0 \quad (4) \end{aligned}}$$

(4) gives $c_4 = 0$.
(3) gives $c_3 = c_4 = 0$
(2) gives $c_2 = c_4 = 0$
(1) gives $c_1 = -c_2 = -0 = 0$

Thus the only solution to

$$c_1\begin{pmatrix}1&0\\0&1\end{pmatrix} + c_2\begin{pmatrix}1&1\\1&0\end{pmatrix} + c_3\begin{pmatrix}0&0\\0&1\end{pmatrix} + c_4\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

is $c_1 = c_2 = c_3 = c_4 = 0$.

Thus, $\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&-1\\1&0\end{pmatrix}$ are linearly independent and hence are a basis for $M_{2,2}$ as discussed earlier. $\blacksquare$

Ex: Let

$V = M_{2,2}$ ← set of 2×2 matrices

$F = \mathbb{R}$.

Previously we showed that

Standard basis

$$\beta_1 = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

$$\beta_2 = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right]$$

are both ordered bases for $M_{2,2}$.

Let's calculate coordinates with respect to these bases.

Consider the matrix $\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}$

Let's find $\left[\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}\right]_{\beta_1}$.

We have

$$\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 3\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus,

$$\left[\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}\right]_{\beta_1} = \langle 3, 4, 0, 1 \rangle$$

Now let's find $\left[\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}\right]_{\beta_2}$.

We need to solve

$$\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + d\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

This becomes

$$\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} b & b \\ b & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} 0 & -d \\ d & 0 \end{pmatrix}$$

which becomes

$$\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+b & b-d \\ b+d & a+c \end{pmatrix}$$

which becomes

$$\boxed{\begin{aligned} a + b &= 3 \\ b - d &= 4 \\ b + d &= 0 \\ a + c &= 1 \end{aligned}}$$

Let's solve!

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-R_1+R_4 \to R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & -2 \end{array}\right)$$

$$\xrightarrow[R_2+R_4 \to R_4]{-R_2+R_3 \to R_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 2 & -4 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 \Leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & -4 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_4 \to R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

This becomes

$$\boxed{\begin{aligned} a+b \quad\quad\quad &= 3 \\ b \quad -d &= 4 \\ c-d &= 2 \\ d &= -2 \end{aligned}}$$

Thus, $\boxed{d=-2}$

$\boxed{c = 2+d = 0}$

$\boxed{b = 4+d = 4-2=2}$

$\boxed{a = 3-b = 3-2=1}$

Therefore,

$$\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Thus,

$$\left[\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}\right]_{\beta_2} = \langle 1, 2, 0, -2 \rangle$$

HW 7 - Part 2

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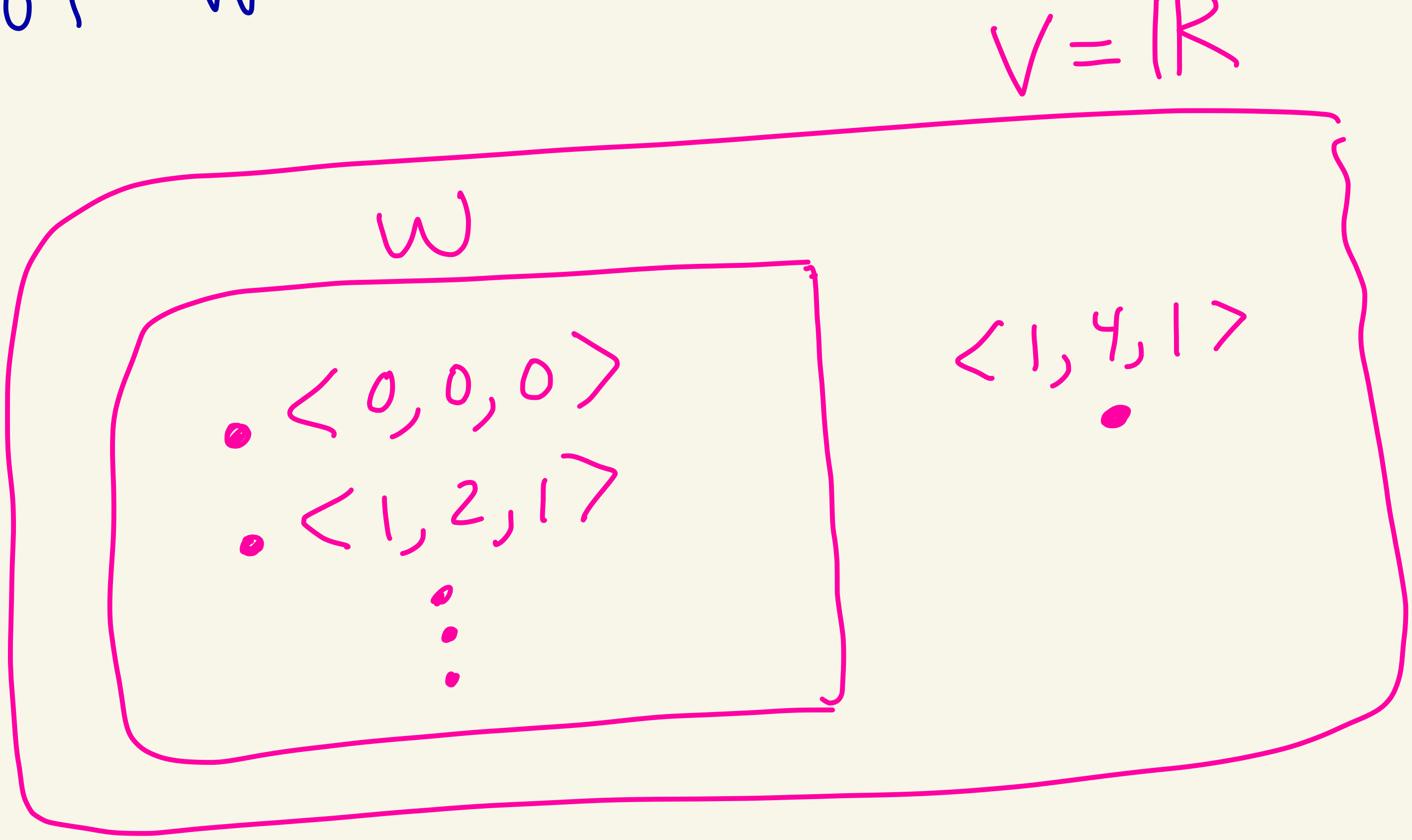
1 (b) In HW 6 we showed that if $V = \mathbb{R}^3$ and $F = \mathbb{R}$ and $W = \{ \langle a, b, c \rangle \mid b = a + c \text{ and } a, b, c \in \mathbb{R} \}$ then W is a subspace of $V = \mathbb{R}^3$.

We will now find the dimension of W and a basis for W.



Let's find a basis for W.

Suppose $\langle a, b, c \rangle$ is in W.

Then, $b = a + c$.

So,

$$
\begin{aligned}
\langle a, b, c \rangle &= \langle a, a+c, c \rangle \\
&= \langle a, a, 0 \rangle + \langle 0, c, c \rangle \\
&= a \langle 1, 1, 0 \rangle + c \langle 0, 1, 1 \rangle
\end{aligned}
$$

Note that $\langle 1, 1, 0 \rangle$ and $\langle 0, 1, 1 \rangle$ are in W.

From the above we see that W is spanned by $\langle 1, 1, 0 \rangle$ and $\langle 0, 1, 1 \rangle$.

Now let's show

$$\langle 1,1,0\rangle, \langle 0,1,1\rangle$$

are linearly independent.

Consider

$$c_1\langle 1,1,0\rangle + c_2\langle 0,1,1\rangle = \underbrace{\langle 0,0,0\rangle}_{\vec{0}}$$

This becomes

$$\langle c_1, c_1, 0\rangle + \langle 0, c_2, c_2\rangle = \langle 0,0,0\rangle$$

which gives

$$\langle c_1, c_1+c_2, c_2\rangle = \langle 0,0,0\rangle.$$

This gives

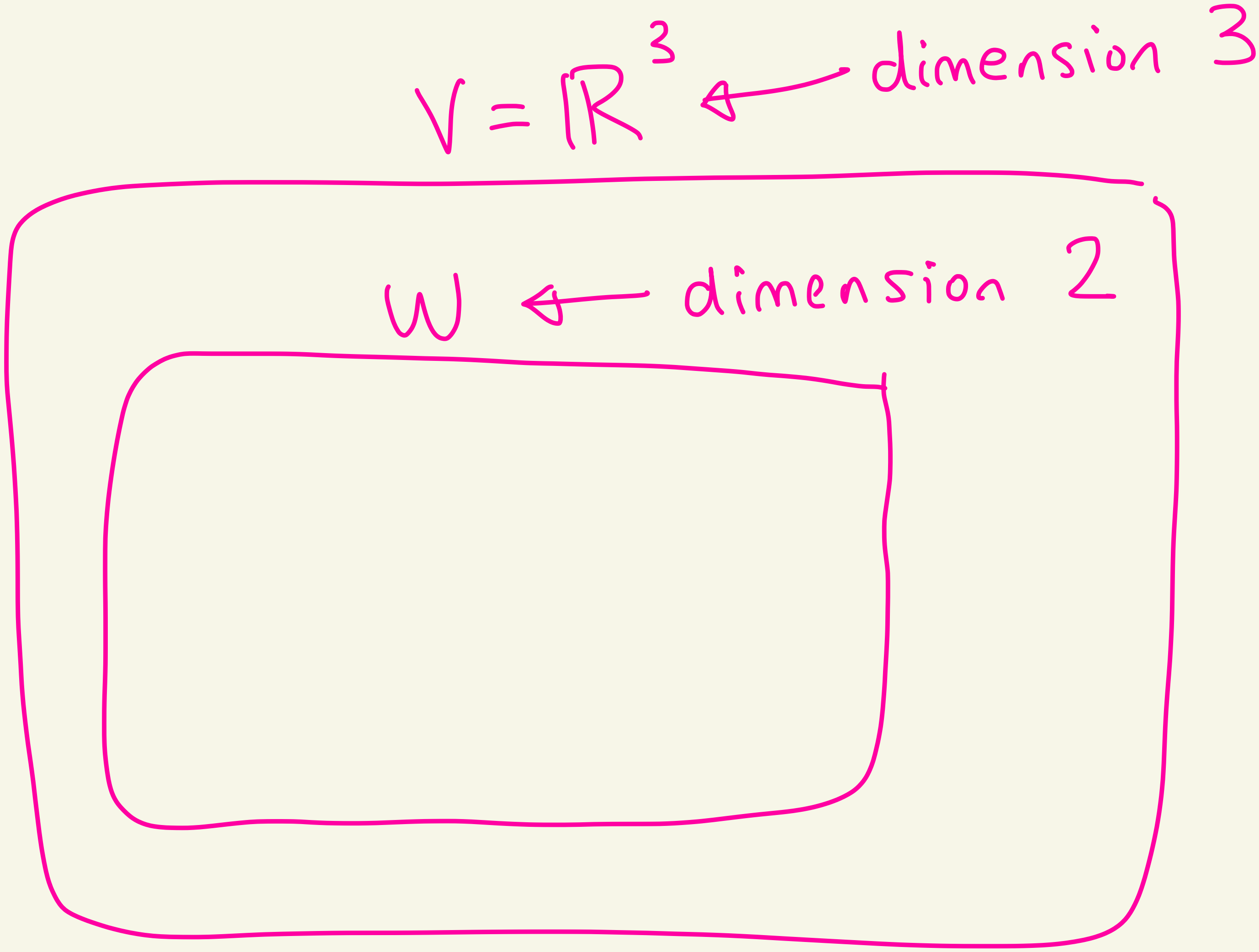
$$\boxed{\begin{aligned} c_1 \quad\quad &= 0 \\ c_1 + c_2 &= 0 \\ c_2 &= 0 \end{aligned}}$$

The only solution is $c_1 = c_2 = 0$.

This means $\langle 1,1,0\rangle, \langle 0,1,1\rangle$ are linearly independent.

Thus, $\langle 1,1,0 \rangle$, $\langle 0,1,1 \rangle$ form a basis for W.

Therefore, the dimension of W is 2 [# elements in basis].



In general we have:

Theorem: Let V be a $\underbrace{\text{finite-dimensional}}$ vector space over a field F.

means: has a basis of finite size

Let W be a subspace of V.

Then:

1. W is finite-dimensional
2. $\dim(W) \leq \dim(V)$
3. If $\dim(W) = \dim(V)$, then $W = V$.
4. If $W = V$, then $\dim(W) = \dim(V)$.